(7 pages) **Reg. No. :** ....................................

**Code No. : 5090** **Sub. Code : HMAM 43**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2020.

Fourth Semester

Mathematics

ANALYTIC NUMBER THEORY

(For those who joined in July 2012-2015)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL the questions.

Choose the correct answer :

1. Which one of the following is not a prime number

   (a) 73 (b) 89

   (c) 17 (d) 119

2. If $(a, b) = 1$ , then $(a + b,\ a - b)$ is

   (a) 1 (b) 2

   (c) 1 or 2 (d) 3

3. The value of $\mu(10) + \phi(10)$ is

(a) 5 (b) 10

(c) 4 (d) 20

4. $\sum_{d/n} \wedge(d)$ is

(a) $\left[\frac{1}{n}\right]$ (b) $\log n$

(c) $n$ (d) 1 or 0

5. If $f$ is multiplicative, then $\prod_{p/n}(1 - f(p))$ is

(a) 0 (b) $\sum_{d/n} \mu(d)\, f(d)$

(c) $\sum_{d/n} \wedge(d)\, f(d)$ (d) $\sum_{d/n} \lambda(d)\, f(d)$

6. If $\alpha$ has a Dirichlet inverse $\alpha^{-1}$, then the equation $G(x) = \sum_{n \le x} \alpha(n)\ F\left(\frac{x}{n}\right)$ implies

(a) $F(x) = \sum_{n \le x} \alpha(n)\, G\left(\frac{x}{n}\right)$

(b) $F(x) = \sum_{n \le x} \alpha^{-1}(n)\, G(x)$

(c) $F(x) = \sum_{n \le x} \alpha^{-1}(n)\, G\left(\frac{x}{n}\right)$

(d) $F(x) = \sum_{n \le x} \alpha^{-1}(n)\, G^{-1}\left(\frac{x}{n}\right)$

7. The average order of $d(n)$ is

(a) $n$ (b) $\log n$

(c) $\frac{\pi^2 n}{12}$ (d) $\frac{3n}{\pi^2}$

8. $\sum_{n \le x} \wedge(n)\left[\frac{x}{n}\right]$ is

(a) $\log[x]$ (b) $\log x!$

(c) $\log[x]$ (d) 1

9. Chebyshev's $\psi$-function is defined by

(a) $\psi(x) = \sum_{n \le x} \lambda(n)$ (b) $\psi(x) = \sum_{n \le x} \wedge(n)$

(c) $\psi(x) = \sum_{n \le x} \log p$ (d) $\psi(x) = \sum_{n \le x} \theta(n)$

10. $\pi(11.62)$ is

(a) 5 (b) 4

(c) 6 (d) 11

PART B — ($5 \times 5 = 25$ marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Prove that there are infinitely many prime numbers.

Or

(b) Prove that if $2^n - 1$ is prime, then $n$ is prime.

12. (a) If $n \geq 1$, prove that $\sum_{d/n} \mu(d) = \left[\frac{1}{n}\right]$.

Or

(b) State and prove the Mobius inversion formula.

13. (a) If $f$ and $g$ are multiplicative, prove that their Dirichlet product $f * g$ is also multiplicative.

Or

(b) Define the Liouville's function $x(n)$ and find a formula for $\sum_{d/n} \lambda(d)$ for $n \geq 1$.

14. (a) If $x \geq 1$, prove that $\sum_{n \leq x} \frac{1}{n} = \log x + C + 0\left(\frac{1}{x}\right)$.

Or

(b) Prove that the average order of $\phi(n)$ is $\frac{3n}{\pi^2}$.

15. (a) Prove that $\lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1$ implies $\lim_{x \to \infty} \frac{\pi(x) \log \pi(x)}{x} = 1$

Or

(b) State and prove Abel's identify.

PART C — ($5 \times 8 = 40$ marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Prove that the infinite series $\sum_{n=1}^{\infty} \frac{1}{p_n}$ diverges.

Or

(b) State and prove the division algorithm.

17. (a) Define the Euler Quotient function $\phi(n)$ and show that for $n \geq 1$ , $\sum_{d/n} \phi(d) = n$ .

Or

(b) For $n \geq 1$, prove that $\phi(n) = n \underset{p/n}{\pi}\left(1 - \frac{1}{p}\right)$.

18. (a) If both $g$ and $f * g$ are multiplicative, prove that $f$ is also multiplicative.

Or

(b) Assume $f$ is multiplicative. Prove that

(i) $f^{-1}(n) = \mu(n)\, f(n)$ for every square free $n$

(ii) $f^{-1}(p^2) = f(p)^2 - f(p^2)$ for every prime $p$.

19. (a) State and prove the Euler's summation formula.

Or

(b) Define the density of the lattice points visible from the origin and find the density of the set of lattice points visible from the origin.

20. (a) Prove that the following relations are logically equivalent :

(i) $\underset{x \to \infty}{lt} \dfrac{\pi(x) \log x}{x} = 1$

(ii) $\underset{x \to \infty}{lt} \dfrac{Q(x)}{x} = 1$

(iii) $\underset{x \to \infty}{lt} \dfrac{\psi(x)}{x} = 1$ .

Or

(b) For every integer $n \geq 2$, prove that $\pi(n) > \dfrac{1}{6} \cdot \dfrac{n}{\log n}$ .

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